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# Book Review: *Percolation*

**Percolation.** Geoffrey R. Grimmett, Springer-Verlag, New York, 1989.

Geoffrey Grimmett's book meets a major need in the development of the theory of percolation. It is the only book available at the moment presenting the mathematical theory of percolation in its full development. It includes most of the major results on the subject, written in a vivid and unified language, with many proofs reworked in the interests of clarity. For these reasons it can be used both as a reference for researchers and as an introduction to percolation.

Appearing 8 years after Kesten's first general account, *Percolation for Mathematicians*, Grimmett's book includes most of the many new developments of recent years: these are the basis of a satisfactory theory for regular graphs in general dimensions, while at the time of the first book on the subject results were well developed only for two-dimensional graphs.

The theory of percolation considered here is essentially mathematical. It deals with random subgraphs of a given graph with countably many vertices. The standard terminology of this subject is readily summarized. Maximal connected components of the subgraph are called clusters and percolation is said to occur if the subgraph contains a cluster with infinitely many vertices (with nonzero probability). The original graph and the random mechanism generating the subgraph determine the essential features of the model. Historically, one of the first models was constructed by taking the $d$-dimensional set of integers $Z^d$ with nearest neighbor bonds; each bond is then assigned to the subgraph with some fixed probability $p$, independently of the other bonds; vertices to which the chosen bonds are incident complete the subgraph. This model, known as the independent bond percolation model in $Z^d$, is appealing both because it is elementary and because functions of the parameter $p$ can be viewed as analogues of more complicated quantities of interest in statistical mechanics. For instance the probability $\theta(p)$ that a fixed vertex belongs to an infinite cluster of the subgraph undergoes a phase transition: there is a critical point $p_c(d)$ such that $\theta(p)$ is identically equal to zero for $p < p_c(d)$ and is nonzero for $p > p_c(d)$. This is reminiscent of a similar transition for the



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spontaneous magnetization in ferromagnetic models. Grimmett's book restricts the discussion essentially to this model, the independent bond percolation model in $Z^d$, since it already poses the most interesting questions in the theory. This also avoids the complications present in discussing other graphs, allowing a uniform and straightforward presentation of the subject. The interested reader can either try to figure out the (often simple) extension to independent percolation models on a general regular graph or, for difficult cases and at least in two dimensions, can refer to Kesten's book. As for other models, several which have various interconnections with percolation are presented in the last chapter, enhancing the flavor of the theory. A systematic treatment of the theory of percolation for dependent models has not been attempted in the book, but this theory is still in an early stage and would require a separate account.

Ancillary use of the independent percolation model on the binary tree is made in discussing the critical exponents. These exponents are thought to be constants, depending only on very general features of the graph, such as the dimension; they are supposed to describe the behavior of various functions of the parameter $p$, such as $\theta(p)$, for $p$ close to the critical point. Computations can be made explicitly when the graph is a regular tree. It is a major challenge in the mathematical theory of percolation to reproduce analogous calculations for $Z^d$. A stimulating heuristic discussion and the rigorous computations for the regular trees are carried out in Chapters 7 and 8 of Grimmett's book. This treatment partly suffers from missing a recent development, only briefly announced in a note added in proof: T. Hara and G. Slade provided a rigorous proof that the values of critical exponents are those predicted by renormalization group theory, at least in high dimensions (on the order of $d > 100$), thereby performing some of the above-mentioned calculations.

The vivid style, the unified language, and the straightforward presentation allow the book to be used for learning about percolation from the beginning. It could also be used as a text at the U.S. graduate level, although it does not contain exercises. The mathematics, as typical of the theory, is fairly elementary, but quite technical. In the book, the prerequisites are not well identified, on the other hand only a general knowledge of some probabilistic tools is required. At the beginning a few pages stimulate the reader's intuition about the subject, as far as physical background and types of questions with which the theory is concerned. But the discussion is cut short before the mathematics starts, and the reader interested in a more intuitive presentation might profit by having a look at one of the books Grimmett recommends as a general introduction or at the last chapter of *Percolation*. For a comprehensive survey of the rigorous results one is then advised to return to the first chapter of Grimmett's book.

*Percolation* is a valuable tool for research both in percolation and in related fields. It contains references to the slightly different terminologies sometimes adopted by various authors and it further presents proofs in different versions, when possible. Finally, intuitive versions of the proofs are often presented before examining the technicalities, as greatly needed, at least in percolation.

Regrettably for the completeness of the book, another new development has already taken place. Soon after its publication, a paper by G. Grimmett and J. Marstrand, "The supercritical phase of percolation is well behaved," solved one of the main problems of the theory: the critical point is the same for independent percolation on the full space $Z^d$ and on the half-space $Z^{d-1} \times Z^+$ or, equivalently, the critical points $P_c(d, k)$ of independent percolation on layers $Z^{d-1} \times \{-k,..., k\}$ converge to $p_c(d)$ as $k$ diverges. The unavailability of this result forced unnatural assumptions all over the book. Thus, the reader now has to bear in mind, while reading through the many places where this is quoted, that these results have been proven. Many other results about the supercritical phase $[p > p_c(d)]$ are presented in the book: uniqueness of the infinite cluster as well as descriptions of the geometry of both infinite cluster and finite clusters. These results completed by the above-mentioned equality of the critical points provide a satisfactory description of the supercritical phase. A similar description of the subcritical phase $[p < p_c(d)]$ is achieved by various results culminating in the proofs that the mean cluster size diverges at $p_c(d)$.

All over the book various unsolved questions make their appearance, of these the conjecture that $\theta(p_c(d)) = 0$ is probably the most popular.

As a pleasant finale to each chapter a set of historical notes revives the romance of the various discoveries and provides a guideline through the already vast literature on the subject.

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